Electron Leakage Through Magnetic Cusps in the Polywell Confinement Geometry

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1. Introduction and Background

Recently, Maffei[1] has reported results of Monte Carlo computer calculations of the trajectories of electrons of fixed energy in the magnetic cusp geometry of a truncated cube Polywell device. These showed large loss rates, much greater than estimated[2] on the basis of the usual theory of cusp loss. In this note we perform simple calculations which explain the results of the Monte Carlo calculation.

It is important to recognize the limited relevance of the Monte Carlo single particle calculations to the Polywell confinement concept; cusp confinement is dominated by collective effects (Berkowitz, *et al.*, Proceedings 2nd International Conference on the Peaceful Uses of Atomic Energy, 1958). The configuration is unstable to inward collapse, and particles which in a single particle picture would be lost by adiabatic invariant arguments in fact are reflected inwards by the magnetic field as modified by the collective effects of the confined high β plasma. Even in terms of single particle loss, the basic problem addressed by the Monte Carlo calculations neglected internal electric fields.

However, since these computer simulations are applicable to early time behavior of the Polywell experiments, when a low density electron gas may be the dominant component of the plasma, it is instructive to examine them in the light of a model of a single particle loss which is developed in the body of this note; the model can include some phenomena neglected in the Monte Carlo calculation, such as the effect of internal electric potential wells.

The approach taken here is to proceed from a relatively simple model to a more complex description, examining each one in turn. The first model relates to the computer calculations of Maffel,[1] in which electrons of constant energy are reflected by cusp mirror magnetic fields on the faces of a truncated cube, but no electric field is present. The analytic model invoked here for this system analyzes a single face cusp of a truncated cube configuration as representative of the complete polyhedral pattern. Electron reflection coefficients, losses and other features and characteristics of this single-face model are found to give good agreement with the results of the Monte Carlo calculation;[1] conditions for low loss rates are indicated. These analyses are all limited to single particle behavior, the basic cusp "mirror reflection" (MR) mode of electron confinement.

These results are then compared with the much lower losses generally expected in a cusp filled with plasma, where collective effects determine the field structure and the loss rates. This case can be visualized by treating the cusp as a perforated sphere; we call the result the "wiffle ball" (WB) model. Finally, we include a radial varying electric field to calculate single particle confinement in combined magnetic and electric geometries, in comparison with the simple MR and WB leakage models used earlier.

2. The Wiffle Ball (WB) Model of Cusp Confinement

In the standard theory of cusp confinement, the collective effect of the confined high β plasma excludes the magnetic field from the interior of the cusp, with a well defined sheath of β = 1 and B = constant separating the higher field exterior from the interior. Particles are assumed to reflect many times from this interface, until they find themselves moving almost parallel to the magnetic field. The particle loss is estimated by a "wiffle-ball" (WB) model, in which particle losses are calculated simply by taking the ratio f_c of the total area of "holes" of gyromagnetic radius r_g on the device quasi-spherical surface of radius R, due to N cusps, to the total surface area of the system. The probability of loss of any one particle that reaches this surface is then simply the ratio

$$f_c = \frac{N}{4}\left(\frac{r_g}{R}\right)^2 \quad (1)$$

and the probability of reflection (instead of loss) is given by the "reflection coefficient"

$$R_w = 1 - f_c \quad (2)$$

The total number of surface collisions n_{refl} by a single particle is the system "current recirculation ratio,"[3] previously defined[4] as G_j, for electrons. Thus

$$n_{ref1} = \frac{1-f_c}{f_c} = G_j = \frac{R_w}{1-R_w} \quad (3)$$

The gyroradius r_{ge} of an electron with mass m_e, velocity v, energy E_k, is:[5]

$$r_{ge} = \frac{m_e vc}{eB} = \frac{\sqrt{2m_e c^2 E_k}}{eB} = \frac{3.37}{B}\sqrt{E_w} \ \text{(cm, Gauss, eV)} \quad (4)$$

which gives the escape fraction as

$$f_c = 2.83\left(\frac{E_w}{B^2}\right)\frac{N}{R^2} \quad (5)$$

3. Mirror Reflection (MR) Model for Single Particle Confinement

Consider classical mirror confinement of a charged particle in a time-independent but slowly space-varying (on a gyroradius space scale) magnetic field of cusp geometry, with no external electric fields. The magnetic moment of a charged particle in a magnetic field is defined as:[6]

$$\mu = \frac{IA}{c} \quad (6)$$

Here I is the current due to gyromagnetic rotation about the field lines, and A is the area bounded by the current path, given by

$$A = \pi r_g^2 \quad (7)$$

Take z as the direction transverse to the B field, and r as the direction along the field. These definitions correspond to the usual conventions employed in the spatial description of the Polywell device, where r is the radial position from the system center, and z is (here) the distance from any given cusp axis. Then, in fields which vary slowly over the gyroradius of the particle, so that

$$r_g v_z \ln(B) << 1 \quad (8)$$

the magnetic moment for singly charged particles ($Z = 1$) becomes

$$\mu = \frac{mv_z^2}{2B} \quad (9)$$

where v_z is the particle speed transverse (perpendicular) to the B field. Now the force acting on a particle to retard its motion towards regions of higher field is just:[7]

$$F = -\mu\nabla_r B = m\left(\frac{dv_r}{dt}\right) \quad (10)$$

where it has been assumed that the variation of B with r is slow. Multiplying by v_r gives

$$\frac{d}{dt}\frac{mv_r^2}{2} = -\mu\frac{dB}{dt} \quad (11)$$

Since the total kinetic energy of each particle is invariant in a collisionless system,

$$E_k = \frac{m}{2}\left(v_r^2 + v_z^2\right) = \text{constant} \quad (12)$$

it is evident from Equations (9), (11), and (12) that

$$\frac{d}{dt}\mu B = \mu\frac{dB}{dt} \quad (13)$$

thus μ is a constant of the motion to lowest order in dB/dr. From this it follows, from Equation (12), that

$$\frac{mv_r^2}{2} = E_k - \mu B \quad (14)$$

so that the particle will be reflected (i.e., v_r will become zero) at the position (r) along the magnetic field at which the field magnitude reaches a value $B(r) = (E_k/\mu) = B_m$. Thus all particles with $(E_k/\mu) < B_m$ will be reflected by the B field and will be "trapped" in the mirror system, so long as μ remains constant.

The reflection coefficient R_m for such trapping of a single particle is just the ratio of particle total kinetic energy to its transverse kinetic energy at $r = r_o$ which is its initial point of entry into the mirror B field at which point the field strength is B_o; thus

$$R_m = \frac{E_k}{E_z(r_0)} \quad (15)$$

This can be determined readily by noting that the transverse energy of the particle is just

$$\frac{mv_z^2}{2} = E_k \sin^2\theta(r) \quad (15b)$$

where $\theta(r)$ is the angle between the total velocity vector and the local B field. Following Spitzer,[7] since

$$\mu = \frac{mv_z^2}{2} B(r) \quad (15c)$$

is a constant of the particle motion, it follows that

$$\mu B(r) = E_k \sin^2\theta(r) \quad (15d)$$

thus

$$\frac{B(r)}{B_0} = \frac{\sin^2\theta(r)}{\sin^2\theta_0} \quad (16)$$

where θ_o is the velocity vector angle of the particle with the field at its initial (low-field) position at $r = r_o$, $B = B_o$. Reflection will occur at $B(r) = B_m$ when

$$\sin^2\theta(r) = 1 \quad (16b)$$

(i.e. when all velocity is transverse), thus the reflection condition is

$$\sin^2\theta_0 > \frac{B_0}{B_m} \quad (16c)$$

for any given particle with initial velocity vector angle θ_o.

Now, the reflection coefficient for a collection of particles with distributed velocity vectors is just the ratio of the number of particles directed towards the high field region, with velocity vectors that satisfy the reflection condition, above, to the total number of particles of all velocity vectors within the mirror field. For isotropic velocity vector distribution, the number of particles per unit solid angle is just $dn = d\Omega$, where the unit solid angle is $d\Omega = 2\pi \sin\theta d\theta$. Thus, the reflection coefficient R for the collection of particles is

$$R = \frac{\int_{\theta_1}^{\theta_{\max}} d\Omega}{\int_{\theta_0}^{\theta_{\max}} d\Omega} \quad (17)$$

where

$$\sin\theta_1 = \sqrt{\frac{B_0}{B_m}} \quad (17b)$$

This reflection coefficient is analogous to that defined in Equations (2)-(3), and is important for assessment of electron leakage in Polywell systems.

Integration of Equation (17) over half-space ($\theta_{max} = \pi/2$) yields, by symmetry, the reflection and loss coefficients for the simple biconic cusp (two cusps) magnetic mirror system as

$$R_2 = \sqrt{1 - \frac{B_0}{B_m}},\ L_2 = 1 - R_2 \quad (18)$$

which is independent of the velocity distribution so long as the velocity vectors are isotropic.

The Polywell systems of interest here are of course not biconic, and the transmission coefficient in Equation (18) must be multiplied by $N/2$, where N is the number of cusps, giving

$$R_N = 1 - L_N;\ L_N = \frac{N}{2}\left[1 - \sqrt{1 - \frac{B_0}{B_m}}\right] \quad (19)$$

In the context of the actual Polywell experiment, we note that cusp loss may in fact be substantially reduced by the use of electrostatic repeller plates in the throat of the cusp. In any such cusp system the particle losses will arise from particles whose velocity vectors lie below the angle θ_1; i.e., those pointed generally along the cusp axis. If the central region around this axis is plugged with a plate charged sufficiently negatively to reflect particles which would otherwise escape, the reflection coefficient will be further increased. Assume that such a "repeller" plate is located at an angular position θ and that a fraction f_r of the particles is reflected by this plate. Then the net reflection coefficient will be modified by addition of a term

$$f_r \int_{\theta_0}^{\theta_1} d\Omega \quad (19b)$$

to the numerator of Equation (17). With this, and limiting integration to θ_{max} as given by Equation (19), the net reflection coefficient including repellers would be

$$R = 1 - (1 - f_r)\frac{N}{2}\left[1 - \sqrt{1 - \frac{B_0}{B_m}}\right] \quad (20)$$

In all of these calculations of R and L it is assumed that the number of cusps N is sufficiently small that their solid angle does not enter the loss cone of the adjacent cusp, i.e., that R and L are less than one.

In order to make use of these results, it is necessary to know the value of B_o, the minimum field from which the particles start their (radial) adiabatic path towards the maximum cusp field. In geometry of the Polywell concept, the field modulus tends to vary as $(r/R)^{mb}$ where mb = 2 for tetrahedral fields, mb = 3 for a truncated cube system, mb = 4 for the next higher polyhedral configuration, etc.[8] Taking

$$B(r) = B_m \left(\frac{r}{R} \right)^{mb} \quad (21)$$

in Equation (20) and assuming that $B_m >> B_o$ gives the mirror reflection model result

$$R_{MR} = 1 - \left(\frac{N}{4} \right) \left(\frac{r_0}{R} \right)^{mb} \quad (22)$$

where $B(r) = B_o$ at $r = r_o$ and fr is neglected hereafter. Here r_o is the minimum radial position at which this model can be thought to apply; i.e., for which the adiabaticity condition (see Equations 8, 9) is satisfied so that μ = constant. A condition for this is that the gyroradius of the particle calculated from the field at r_o be less than

$$L_B = \frac{1}{\nabla_r \ln B} \quad (22b)$$

This condition yields

$$\left(\frac{r_0}{R} \right)^{mb+1} = \frac{r_g}{mbR} \quad (23)$$

where

$$r_g = \frac{v_e}{\left(eB_m / mc \right)} \quad (23b)$$

The loss rate is

$$L_{MR} = \frac{N}{4} \left(\frac{1}{mb} \frac{r_g}{R} \right)^{\frac{mb}{1+mb}} \quad (24)$$

4. Comparison of Wiffle Ball (WB) and Mirror Reflection (MR) Losses

The leaky-perforated-sphere or "wiffle-ball" (WB) model of the system has been discussed above. Various earlier experimental and theoretical studies[9,10,11] have supported this model for the conditions expected in the real system, where collective effects dominate confinement. Losses in the WB model result in a reflection coefficient, from Equations (1)-(2) of

$$R_{WB} = 1 - \frac{N}{4} \left(\frac{r_g}{R} \right)^2 \quad (25)$$

and a loss rate

$$L_{WB} = \frac{N}{4} \left(\frac{r_g}{R} \right)^2 \quad (26)$$

where r_g is the electron gyroradius in the cusp (maximum) surface field.

To compare the loss from the MR single particle calculation and the WB high β cusp calculation, we calculate the ratio of the loss rate, 1 - R, in Equation (24) to that in Equation (26)

$$\frac{L_{MR}}{L_{WB}} = \left(\frac{R}{r_g} \right)^{\frac{2+mb}{1+mb}} \left(\frac{1}{mb} \right)^{\frac{mb}{1+mb}} \quad (27)$$

Thus classical mirror reflection will always lead to greater losses than with the wiffle ball model, since the gyroradii is less than the system radius (i.e., $r_g/R < 1$).

The computer studies of Maffei,[1] previously mentioned, all showed the behavior outlined above. The agreement between this simple model and the calculations is quite good. This comparison will be described in a separate report.

5. Electric Fields and Recirculation in the MR Model

In an actual system the electrons do not have constant kinetic energy because they are moving in an electrostatic potential well, but their total energy — potential plus kinetic — will remain constant (absent collisions). The MR analyses can be extended to encompass this condition by inclusion of the electric potential ϕ in the expression for total energy, thus

$$E_0 = E_k + q\phi = \frac{m}{2}\left(v_r^2 + v_z^2 \right) + q\phi \\ = \text{constant} = q\phi_0 \left(1 + \alpha \right) \quad (28)$$

replaces Equation (12), previously. Here q is the charge on the particle, and α is a constant which relates the relative size of the maximum potential energy and the total energy. The potential ϕ is defined by

$$\phi(r) = \phi_0 \left[1 - \left(\frac{r}{R} \right)^{me} \right] \quad (29)$$

where ϕ_o is the well depth and is taken of negative sign, and me is the exponent of the power law equation describing the potential radial variation.

Now, the force equation is just that from Equation (10) for mirror reflection, with a term added to account for the force due to the gradient of the electric potential, from Equation (29). Thus,

$$F = -\mu\nabla_r B - q\nabla_r \phi = m\frac{dv_r}{dt} \quad (30)$$

Reducing terms as before, it is easily shown that μ remains a constant of the motion. This is obvious since the addition of an electric field does not affect particle motion transverse to the *B* field lines, but acts only in the radial direction.

The reflection coefficient here can be obtained in the same manner as before (e.g., Equations 16-20), leading to

$$\sin^2\theta_1 = (1+\alpha)\frac{\left(\frac{r_0}{R}\right)^{mb}}{\alpha + \left(\frac{r_0}{R}\right)^{me}} \quad (31)$$

where all electrons originating at r_o with $\theta > \theta_1$ will be trapped by mirror reflection at or before they reach the radial position $r = R$, which leads to

$$R_e = 1 - \frac{N}{4}(1+\alpha)\frac{\left(\frac{r_0}{R}\right)^{mb}}{\alpha + \left(\frac{r_0}{R}\right)^{me}} \quad (32)$$

This reflection coefficient is related to the electron recirculation factor G_j, previously defined,[2] by

$$G_j = \frac{1}{1-R_e} \text{ or } R_e = \frac{G_j - 1}{G_j} \quad (33)$$

Using Equation (32), this becomes

$$G_j = \frac{\alpha + \left(\frac{r_0}{R}\right)^{me}}{(1+\alpha)\left(\frac{r_0}{R}\right)^{mb}}\frac{4}{N} \quad (34)$$

As an example take $m_e = m_b = 3$, $f_r = 0$, $N = 14$, and $\langle r_o \rangle = 1/10$, which represents the present HEPS experimental parameters. Then the electron current "gain" is

$$G_j = \frac{2.8x10^2 a}{1+a} \quad (35)$$

Earlier studies[4,12] determined that G_j of tens of thousands would be required for energy gain in the system. Thus the single particle limit, with any reasonable magnetic field strength, would be much too lossy for a reactor. This is no surprise. Indeed, the reason for the Polywell geometry is because cusp losses are orders of magnitude reduced from MR model losses. It is however worth asking whether single particle losses during *startup* will place too great a strain on the electron injection system. This is calculated in the next section.

6. Electron Current Limits

From the electron recirculation factor (Equation 35), it is possible to estimate the electron current required to balance electron losses in the low density phase of startup, when the MR calculation is presumably valid. To do this, note that the electron lifetime in the system is just the transit time across the device, multiplied by the recirculation factor, thus

$$t_{life} = t_{tran} G_j \quad (36)$$

where t_{tran} is defined as an electron transit time averaged over the electron distribution, typically of order R/v_{inj}.

The rate of loss L_e is the total number of electrons divided by the lifetime, and must be made up by injection of a total electron current I_e, thus

$$L_e = \frac{N_{tot}}{t_{life}} = \frac{4\pi R^3 n_e}{3t_{life}} = \frac{I_e}{q} \quad (37)$$

where n_e is the average electron density. This can be expressed, roughly, in terms of potential well depth from Poisson's equation,

$$e\phi_0 = 2\pi n_e (eR)^2 \quad (38)$$

With this, the average density can be determined from Equation (38) and the injection

current can be written in terms of the injection energy, from Equation (37) together with the definition of α, as

$$I_e = \frac{R\phi_0 10^{-10}}{3t_{tran}G_j(1+\alpha)} \quad (39)$$

for ϕ_o in Volts, current in amps, and G_j taken from Equation (34).

As an example consider the case of maximum required current, where $\alpha = 0$ and $G_j = 1$. Then the maximum electron-only current to maintain the potential well is given by

$$I_e = 3.11x10^{-13}\phi_0 v_{inj} \quad (40)$$

for ϕ_o in Volts and v_{inj} in cm/sec. If $\phi_o = 10^5$ V and $v_{inj} = 1.3 \times 10^{10}$ cm/sec, then the limiting current is $I_e = 404$ amps. This indicates that only a modest current is required during the low density startup phase to balance single particle losses, even without repellers.

In conclusion, we recognize that computer code calculations are essential to help understand these complicated systems. The analyses above show some of the general features of these devices; however, particularly because of their neglect of collective effects, they are only indicative of the behavioral features of the system, and even then only in the parameter range where they apply. It is both significant and encouraging that no critical limitations to machine operation have yet been found from these fundamental analytic studies.

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